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The Continued Fraction of e

While the decimal expansion of the Euler's constant $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$ is not periodic, it also does not seem to exhibit any clear patterns. In stark contrast to that, the simple continued fraction of e, while also not periodic, has a very intriguing pattern: $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, ...]$. The purpose of this note is to prove this identity, or more precisely to provide details to a proof published in 1737 by Leonhard Euler.

The simple continued fraction expansion of e starts as $[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, ...]$. There is a pattern evident in the expansion: [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, . . .] which continues for the rest of the infinite continued fraction as will be shown later.

We will first write this continued fraction in a slightly different form:

$$[1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, ...] \tag{1}$$

in order to make the pattern evident throughout the list notation. Note that although we usually write the terms of the simple continued fraction expansion as integers greater than 0, we temporarily relax that condition here in order to define a relationship between the partial convergents of (1).

Notice that the following relationships hold for the terms of (1), for any integer $n \geq 0$:

$a_{3n} = 1$,

$a_{3n-1} = 1$, and

$a_{3n-2} = 2(n - 1)$, where the terms defined by a_{3n-2} are an arithmetic subsequence of (1) written explicitly.

For any continued fraction $[a_0, a_1, a_2, ...]$ we can compute the ith partial convergent $\frac{p_i}{q_i}$ where p_i and q_i can be calculated recursively using the following equations:

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \tag{2}$$

$$\text{where } p_{-1} = 1 \,,\ p_0 = a_0 \,,\ q_{-1} = 0 \,,\ q_0 = 1.$$

For the continued fraction expansion of (1), we compute several partial convergents below:

i	0	1	2	3	4	5	6	7	8	9	10	11
p_i	1	1	2	3	8	11	19	87	106	193	1264	1457
q_i	1	0	1	1	3	4	7	32	39	71	465	536
$\frac{p_i}{q_i}$	$\frac{1}{1}$	d.n.e	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{8}{3}$	$\frac{11}{4}$	$\frac{19}{7}$	$\frac{87}{32}$	$\frac{106}{39}$	$\frac{193}{71}$	$\frac{1264}{465}$	$\frac{1456}{536}$

Upon examining the table above, we observe that p_i and q_i appear to satisfy the following recurrence relation, where r_i is a stand in for p_i or q_i:

$$r_{3n} = r_{3n-1} + r_{3n-2},$$
$$r_{3n-1} = r_{3n-2} + r_{3n-3},$$
$$r_{3n-2} = 2(n-1)r_{3n-3} + r_{3n-4}.$$

We can prove this observation by (2):

We know that for all a_{3n} s.t. $n \in \mathbb{Z}^+ \cup \{0\}$, $a_{3n} = 1$, and $a_{3n-1} = 1$. By substituting into (2), we get

$$r_{3n} = r_{3n-1} + r_{3n-2},$$

$$r_{3n-1} = r_{3n-2} + r_{3n-3}.$$

Since $a_{3n-2} = 2(n-1)$, it follows that

$$r_{3n-2} = 2(n-1)r_{3n-3} + r_{3n-4}.$$

Therefore, this relation holds for $n \in \mathbb{Z}^+ \cup \{0\}$.

If we can somehow use this recurrence in order to compute r_{3n} from r_{3n-3}, r_{3n-3} from $r_{3n-6}$and so on, we would be able to compute the sixth partial convergent from the third, the ninth from the sixth, and so on. This subsequence of partial convergents obtained from calculating every three convergents converges to the same real number as the entire sequence of partial convergents. Therefore, our goal will be to collapse the recurrence relation found above into one recurrence that uses r_{3n}, r_{3n-3}, and r_{3n-6} only.

We have:

$$r_{3n} = r_{3n-1} + r_{3n-2}$$

$$
\begin{aligned}
&= r_{3n-2} + r_{3n-3} + r_{3n-2} \\
&= 2r_{3n-2} + r_{3n-3} \\
&= 2[(2n - 2)r_{3n-3} + r_{3n-4}] + r_{3n-3} \\
&= (4n - 4)r_{3n-3} + 2r_{3n-4} + r_{3n-3} \\
&= (4n - 3)r_{3n-3} + 2r_{3n-4} \qquad (3)
\end{aligned}
$$

Expanding our recurrence relation for the next few iterations:

$$
\begin{aligned}
r_{3n-3} &= r_{3n-4} + r_{3n-5}, \\
r_{3n-4} &= r_{3n-5} + r_{3n-6}.
\end{aligned}
$$

From this we get:

$r_{3n-5} = r_{3n-3} - r_{3n-4}$, which leads to [rearrange $r_{3n-3} = r_{3n-4} + r_{3n-5}$]

$r_{3n-4} = r_{3n-3} - r_{3n-4} + r_{3n-6}$, so we obtain

$r_{3n-4} = r_{3n-5} + r_{3n-6}$

$2r_{3n-4} = r_{3n-3} + r_{3n-6}$ (4)

Combining (3) and (4) we get:

$$
\begin{aligned}
r_{3n} &= (4n - 3)r_{3n-3} + r_{3n-3} + r_{3n-6} \\
&= (4n - 2)r_{3n-3} + r_{3n-6}, \text{ so finally}
\end{aligned}
$$

$$r_{3n} = 2(2n - 1)r_{3n-3} + r_{3n-6} \qquad (5)$$

Set $x_n = p_{3n}$ and $y_n = q_{3n}$ for all $n \geq 0$.

Using the collapsed recurrence relation (5), we can now calculate x_n, y_n, and $\frac{x_n}{y_n}$ for some small values of n:

n	0	1	2	3	4	. . .
x_n	1	3	19	193	2721	. . .
y_n	1	1	7	71	1001	. . .
$\frac{x_n}{y_n}$	1	3	2.714…	2.71830…	2.7182817…	. . .

Notice that the terms of the sequences x_n, y_n follow the recurrence defined below, where z_n is a stand in for x_n or y_n.

$$z_n = 2(2n - 1)z_{n-1} + z_{n-2} \text{ for all } n \geq 2. \tag{6}$$

Integral Sequences

Our goal now is to show that the limit of the partial convergents (i.e. the limit of the sequence of $\frac{x_n}{y_n}$ as *n* approaches infinity) approaches *e*.

In order to show that $\lim_{n \to \infty} \frac{x_n}{y_n} = e$, we will define a sequence of real numbers T_0, T_1, T_2.... by the following integrals:

$$T_n = \int_0^1 \frac{t^n(t-1)^n}{n!} e^t dt.$$

In order to get a better idea of what these terms are, we compute the first two terms of the sequence T_n, namely T_0 and T_1.

Evaluating T_0 we get:

$$T_0 = \int_0^1 e^t dt = e^t\Big|_0^1 = e - 1.$$

Now we compute T_1, where:

$$T_1 = \int_0^1 t(t - 1)e^t dt.$$

To evaluate this integral we will need to do integration by parts twice. For the first integration by parts, let $u = t(t - 1)$ and $dv = e^t dt$. Then, $v = e^t$ and $du = (2t - 1)dt$. So, we get

$$T_1 = t(t - 1)e^t\Big|_0^1 - \int_0^1 e^t(2t - 1)dt.$$

Since the integral is evaluated from 0 to 1, uv will evaluate to 0. So,

$$T_1 = - \int_0^1 e^t(2t - 1)dt.$$

Using integration by parts again, setting $u = 2t - 1$ and $dv = e^t dt$, we get $v = e^t$ and $du = 2dt$. We now calculate:

$$\begin{aligned}
T_1 &= -\left[(2t-1)e^t\Big|_0^1 - \int_0^1 2e^t dt\right] \\
&= -\left[(2\cdot 1 - 1)e^1 - (2\cdot 0 - 1)e^0 - 2e^t\Big|_0^1\right] \\
&= -\left[e + 1 - (2e^1 - 2e^0)\right] \\
&= -\left[e + 1 - 2e + 2)\right] \\
&= -e - 1 + 2e - 2. \text{ So,} \\
T_1 &= e - 3
\end{aligned}$$

Looking closely at these terms, we can see that $T_0 = y_0 e - x_0$ since $x_0 = 1$ and $y_0 = 1$, and $T_1 = y_1 e - x_1$ since $x_1 = 3$ and $y_1 = 1$. If this relationship, $T_n = y_n e - x_n$, holds for all n, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} (e - \frac{T_n}{y_n})$. It would then suffice to show that $\lim_{n\to\infty} T_n = 0$ to prove the continued fraction expansion does in fact converge to e. (Note that $\lim_{n\to\infty} y_n = \infty$).

In order to show that $T_n = y_n e - x_n$ holds for all n, we will first prove that the recurrence relations (6) found for x_n and y_n in fact hold for T_n.

In order to do this, we first need to simplify T_n. We will do so by performing integration by parts twice. Recall that

$$T_n = \int_0^1 \frac{t^n(t-1)^n}{n!} e^t dt.$$

Let $u = \frac{t^n(t-1)^n}{n!}$ and $dv = e^t dt$. Then $v = e^t$ and

$$du = \frac{d}{dt}\left[\frac{t^n(t-1)^n}{n!}\right] = \frac{nt^{n-1}(t-1)^n + t^n n(t-1)^{n-1}}{n!} dt = \frac{t^{n-1}(t-1)^n + t^n(t-1)^{n-1}}{(n-1)!} dt.$$

So,

$$T_n = \frac{t^n(t-1)^n}{n!} e^t\Big|_0^1 - \int_0^1 \frac{t^{n-1}(t-1)^n + t^n(t-1)^{n-1}}{(n-1)!} e^t dt.$$

Notice that $uv\big|_0^1 = \frac{t^n(t-1)^n}{n!} e^t\Big|_0^1$ evaluates to 0 because when $t = 1$, $(t-1)^n = 0$ and when $t = 0$, $t^n = 0$. Thus,

$$T_n = -\int_0^1 \frac{t^{n-1}(t-1)^n + t^n(t-1)^{n-1}}{(n-1)!} e^t dt.$$

Now we will use integration by parts again. Let $u = \frac{t^{n-1}(t-1)^n + t^n(t-1)^{n-1}}{(n-1)!}$ and $dv = e^t dt$. Then $v = e^t$ and

$$du = \frac{d}{dt}\left[\frac{t^{n-1}(t-1)^n + t^n(t-1)^{n-1}}{(n-1)!}\right]$$

$$= \frac{d}{dt}\left[\frac{t^{n-1}(t-1)^n}{(n-1)!}\right] + \frac{d}{dt}\left[\frac{t^n(t-1)^{n-1}}{(n-1)!}\right]$$

$$= \frac{(n-1)t^{n-2}(t-1)^n + n(t-1)^{n-1}t^{n-1}}{(n-1)!} + \frac{nt^{n-1}(t-1)^{n-1} + t^n(n-1)(t-1)^{n-2}}{(n-1)!}$$

$$= \frac{t^{n-2}(t-1)^n + t^n(t-1)^{n-2}}{(n-2)!} + \frac{nt^{n-1}(t-1)^{n-1} + nt^{n-1}(t-1)^{n-1}}{(n-1)!}$$

Notice, when computing $uv\Big|_0^1 = \frac{t^{n-1}(t-1)^n + t^n(t-1)^{n-1}}{(n-1)!} e^t\Big|_0^1$, the expression vanishes as we have seen in the integration above because of the bounds of the integral. So,

$$T_n = \int_0^1 \left(\frac{t^{n-2}(t-1)^n + t^n(t-1)^{n-2}}{(n-2)!} + \frac{nt^{n-1}(t-1)^{n-1} + nt^{n-1}(t-1)^{n-1}}{(n-1)!}\right) e^t dt$$

$$= \int_0^1 \frac{t^{n-2}(t-1)^n + t^n(t-1)^{n-2}}{(n-2)!} e^t dt + 2n\int_0^1 \frac{(t-1)^{n-1}t^{n-1}}{(n-1)!} e^t dt$$

$$= \int_0^1 \frac{t^{n-2}(t-1)^{n-2}}{(n-2)!}\left[(t-1)^2 + t^2\right] e^t dt + 2nT_{n-1}$$

$$= \int_0^1 \frac{t^{n-2}(t-1)^{n-2}}{(n-2)!} e^t \left[2t^2 - 2t + 1\right] dt + 2nT_{n-1}$$

$$= \int_0^1 (2t^2 - 2t) \frac{t^{n-2}(t-1)^{n-2}}{(n-2)!} e^t dt + \int_0^1 \frac{t^{n-2}(t-1)^{n-2}}{(n-2)!} e^t dt + 2nT_{n-1}$$

$$= 2\int_0^1 \frac{t(t-1)t^{n-2}(t-1)^{n-2}}{(n-2)!} e^t dt + T_{n-2} + 2nT_{n-1}$$

$$= 2\int_0^1 \frac{t^{n-1}(t-1)^{n-1}}{(n-2)!} e^t dt + T_{n-2} + 2nT_{n-1}$$

$= \ 2(n-1)T_{n-1} + T_{n-2} + 2nT_{n-1}$. So we finally get that

$$T_n = 2(2n-1)T_{n-1} + T_{n-2}, \tag{7}$$

which is the recurrence we were looking for.

Induction

Now that we have shown that T_n satisfies a recurrence relation (7) similar to (6), we will use induction to prove that

$$T_n = y_n e - x_n \tag{8}$$

holds for all *n*.

As we showed earlier, (8) holds for $n = 0$ & 1. We want to show $T_n = y_n e - x_n$ for $n \geq 2$.

We have our base cases:

$$T_0 = y_0 e - x_0,$$
$$T_1 = y_1 e - x_1.$$

So we can formulate an inductive hypothesis that

$$T_k = y_k e - x_k$$

Holds for some $k \in \mathbb{Z}_+$.

We want to show:

$$T_{k+1} = y_{k+1} e - x_{k+1}$$

Notice, by the recurrence relation (6) we have:

$$y_{k+1} = 2(2(k+1)-1)y_{k+1-1} + y_{k+1-2}$$

$$= (4k+2)y_k + y_{k-1}$$

and $\quad x_{k+1} = (4k+2)x_k + x_{k-1}$. We also know that $T_{k+1} = (4k+2)T_k + T_{k-1}$ by setting $n = k+1$ in (7).

If we can show that $T_{k+1} = ((4k+2)y_k + y_{k-1})e - (4k+2)x_k + x_{k-1}$ we will be done.

We have:

$$((4k + 2)y_k + y_{k-1})e - (4k + 2)x_k + x_{k-1} =$$

$$= (4k + 2)y_k e - (4k + 2)x_k + y_{k-1}e + x_{k-1}$$

$$= (4k + 2)(y_k e - x_k) + y_{k-1}e + x_{k-1}$$

$$= (4k + 2)T_k + T_{k-1}$$

$$= T_{k+1} \text{ by the recurrence relation (7).}$$

This completes the induction, thus, we have shown that $T_n = y_n e - x_n$ holds for all n.

It is left to show that $\lim_{n \to \infty} (e - \frac{T_n}{y_n}) = e$. This can be accomplished by showing that $\lim_{n \to \infty} T_n = 0$.

Notice that we have by definition:

$$\lim_{n \to \infty} T_n = \lim_{n \to \infty} \int_0^1 \frac{t^n (t-1)^n}{n!} e^t dt$$

Upon inspecting the integrand, we can see that the integral cannot exceed $\frac{e}{n!}$. This is because the largest value that t^n can take on is 1. Similarly, the largest value that $(t - 1)^n$ can take on is also 1 and e^t will have a maximum of *e*. We can also observe that the value of the integral will never go below $\frac{-e}{n!}$ as $(t - 1)^n$ (the only possibly negative factor) is never smaller than -1, while $t^n e^t$ is bounded from above by *e*. Thus we have determined that the desired limit must be between the limits of the identified bounds $\frac{-e}{n!}$ and $\frac{e}{n!}$. So,

$$\lim_{n \to \infty} \frac{-e}{n!} \leq \lim_{n \to \infty} T_n \leq \lim_{n \to \infty} \frac{e}{n!}$$

$$0 \leq \lim_{n \to \infty} T_n \leq 0$$

Thus, the limit must evaluate to 0.

Therefore,

$$\lim_{n \to \infty} \frac{x_n}{y_n} = \lim_{n \to \infty} \left(e - \frac{T_n}{y_n}\right) = e$$

Therefore, the sequence of partial convergents $\frac{x_n}{y_n}$ approaches *e*, and thus we have shown that the continued fraction expansion [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, …] does in fact converge to *e*.